Twist and Writhe near Max Turns in a Rubber Motor

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Introduction

Following a well-trodden path, I flew rubber-band-powered balsa airplanes as a kid and didn't think too much about how they worked. I do remember noticing the rows upon rows of knots that formed during winding and dissolved during the power run. More recently I had a rebirth of interest in rubber power when I built a Guillows Cadet for a friend's small son. I got hooked and went on to build and fly a Prairie Bird, a few Cloud Tramps and Flying Aces Moths, and then a Gollywock, a Senator and a Wren. Almost immediately I was confronted by the annoying problem of rubber "bunching up" in the fuselage and ruining the glide balance. This led me to spring stops and Garami clutches, which were fun but didn't definitively solve the balance problem, since the fore-and-aft weight distribution at the stop point was not always reproducible. Curiosity finally led me to two articles in Free Flight Quarterly by Rene Bahout reporting on a clever series of rubber motor experiments he conducted in France during the 1940's, which revealed surprisingly large fluctuations in torque and tension accompanying the dissolution of rows of knots. This more or less explained the irreproducibility of the spring stop point and motivated me to dig deeper.

My investigation revealed that the complex twisting and kinking patterns that accompany the winding and unwinding of rubber model airplane motors also occur in other situations as far a field as DNA coiling, solar plasma physics and motions of undersea cables, and have been studied mathematically for several decades. In mathematical terms, Tw represents twist and what we modellers call kinking or knotting is referred to as writhe or Wr. A third quantity, linking number Lk, was introduced by Romanian scientist Gheorghe Calugareanu in 1961 (1) along with the formula

$$Lk = Tw + Wr \qquad (1)$$

which is known as the *Calugareanu Invariant*. Later, Renzo Ricca in a 1995 paper (2) analyzed the elastic equilibrium between twist and writhe that leads to the twist-writhe transformations seen during winding and unwinding of elastic strings like rubber bands.

For a rubber band motor, Calugareanu's formula says that the total turns put in by the winder, Lk, are distributed between twist rotations Tw and writhe kinks Wr, while Ricca's analysis shows that the distortion of the rubber will exchange between these two modes to minimize the total elastic energy. Experience shows that when a partially wound rubber motor is stretched, the kinks transform into twist, while as it is relaxed the motor untwists as writhe kinks form, as illustrated for the simplest case in Figures 1 a to c.

The following paragraphs will use Calugareanu's concept, along with ideas from Ricca's paper, and a very simple geometrical model to investigate the structure of a wound rubber motor near the breaking point, aka "max turns."

Model

An idealized model of a wound motor is shown in Figure 2. A cylindrical strand of radius R and length L is stretched to maximum extension xL and wound on a zero-radius spool of length L. The winding turns are completely incorporated as writhe loops; i.e., there is no twist in the wound structure. In the terms of Equation 1, $Tw = 0$ and $Lk = Wr$.

Since the volume of the rubber motor is unchanged by stretching (i.e., the Poisson's ratio of rubber is 0.5)

$$\pi R^2 L = xLa \qquad (2)$$

and the cross-sectional area of the stretched motor a is

$$a = \pi R^2/x \qquad (3)$$

Assuming that the circumferential turns in the wound motor are tightly packed, the maximum number of turns

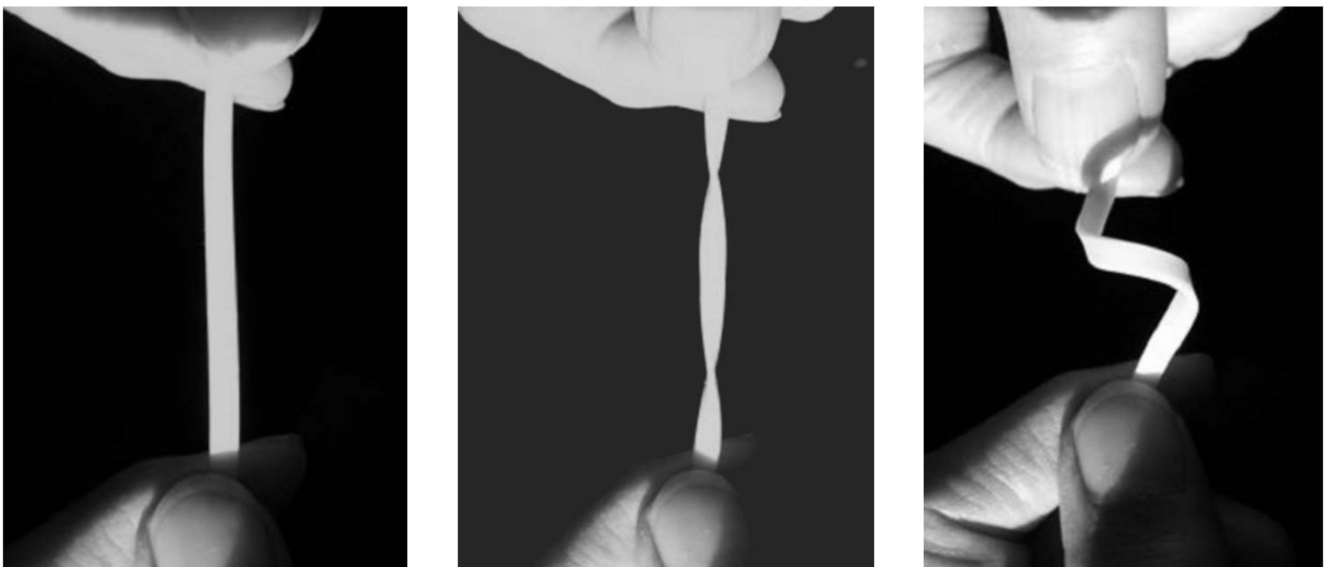

Fig. 1 Interchange of the deformation of a rubber strip between twist and writhe modes

Fig 1a Lk = 0 Fig. 1b Lk = Tw = 1, Wr = 0 Fig. 1c. Lk = Wr = 1, Tw = 0

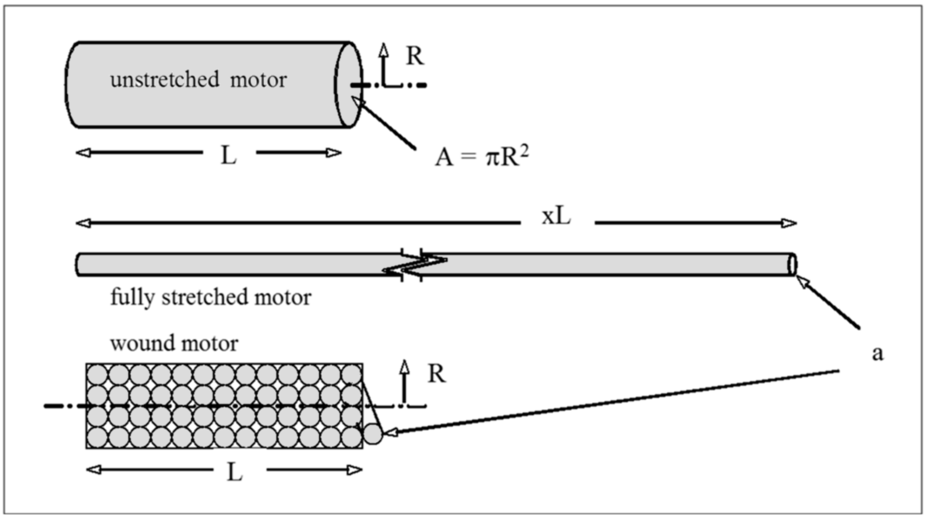

Fig. 2 Geometrical model of wound rubber motor

Nmax can be approximately calculated by noting the crossings of the turns (of cross section a) with the diamterical longitudinal section of area 2RL. As each turn crosses the longituninal cross section twice, it follows that:

$$Nmax = Lk = RLx/\pi R^2 = xL/\pi R \qquad (4)$$

On each "slice" of the wound motor the turns vary in raius from 0 to R, with an average of R/2. Thus the total number of turns can be computed also by dividing the total length at maximum extension xL by the mean wrapping circumference πR, a method that also leads to the result of Eq. 4.

John Barker (Ref. 4) advanced a similar total-length/circumference calculation of max turns in a 1995 article in Free Flight News. Analysis of Barker's somewhat different results and interpretation is beyond the scope of this note.

Discussion

For a real motor made from n strands of rubber of width w and thickness t, the unstretched cross-sectional area πR^2 is equal to nwt, so the equivalent radius is

$$R = (nwt/\pi)^{1/2} \qquad (5)$$

Substituting this result into Eq. 4 and simplifying gives the following expression for max turns per unit length

$$Nmax/L = x/(\pi nwt)^{1/2} \qquad (6)$$

Figure 3 shows a plot of $Nmax /L$ vs. n for motors made from 0.125" wide 0.0425" thick rubber calculated using Equation 6 for an extension ratio x= 10, and compared with test data on 06/09 Tan Super Sport reported by Mark Bennett on the Hippocket Aeronautics website, and typical F1B motor data supplied by Carrol Allen. The plot also includes my data on 6-strand 1/8 in. motors made from 06/07 Tan SS.

As the figure shows, the $Tw = 0$ model correctly predicts the $n^{-1/2}$ functional dependence for max turns vs. number of strands but the predicted magnitude is approximately half that observed experimentally. Is the factor-of-two discrepancy just a coincidence or is it a clue pointing to the structure of wound rubber motors near max turns?

Ricca's paper shows that twisted elastic strings in general minimize their elastic energy by partitioning Lk (i.e., turns) between twist and writhe, and that the equilibrium ratio between twist and writhe depends on Lk. This is consistent with common experience with winding rubber motors wherein the early turns at a given extension are completely taken up by twist. Writhe kinks begin to form as more and more turns are added. At any stage of winding writhe kinks can be partially converted back to twist by stretching the motor.

A straightforward way to account for the factor of two discrepancy would be to suggest that the simple geo-

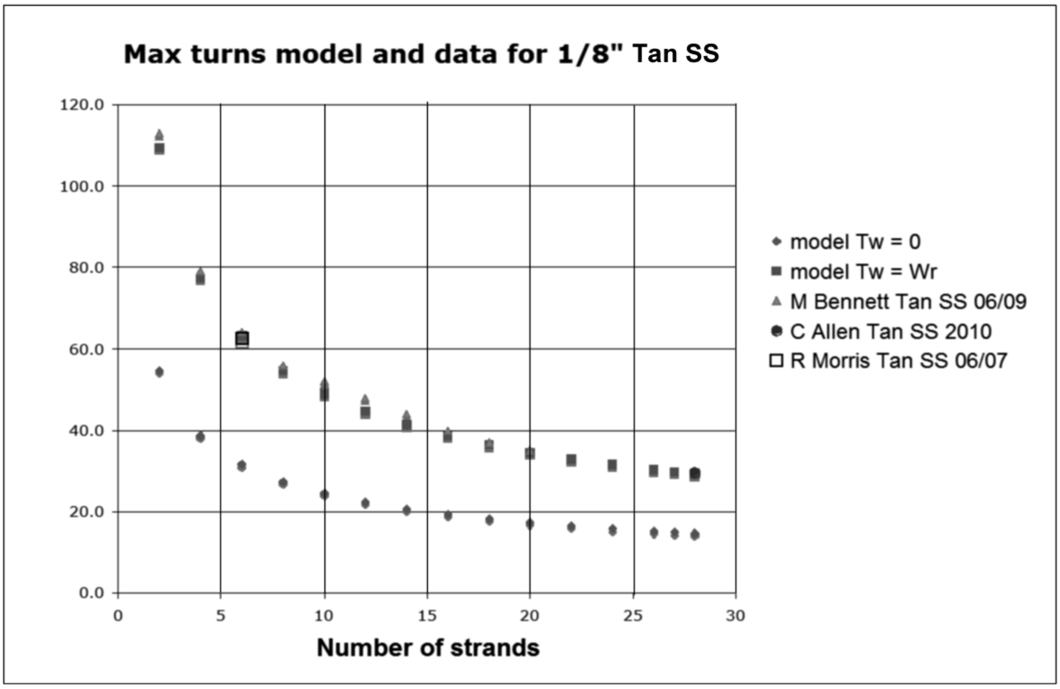

Figure 3 Comparison of model prediction with data for 1/8" FAI Tan Super Sport rubber

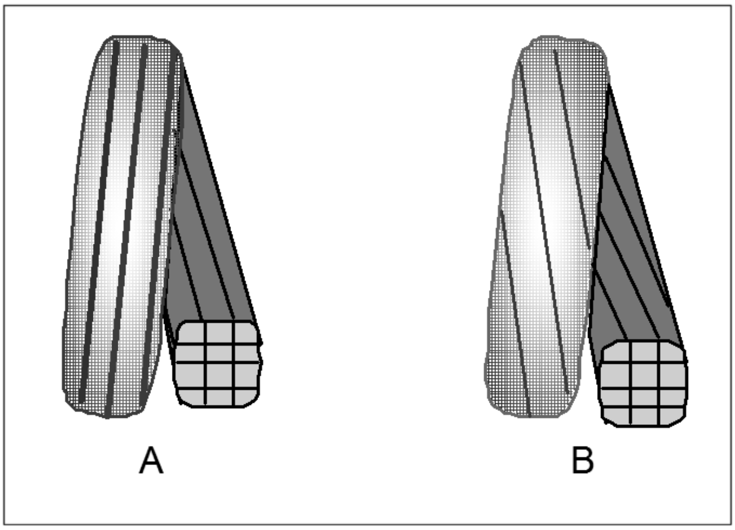

Fig. 4 Schematic of two wound rubber motors: A. with Tw = 0, and B. with Tw = Wr

metrical model accurately predicts writhe turns for the maximally wound motor, but that the structure also contains a twist component. The factor of two is neatly accounted for, if

Tw = Wr, as shown by the plot in Figure 3 and illustrated schematically in Figure 4B. The suggested 1:1 twist-writhe distribution is consistent with the following commonly quoted rule-of-thumb process for winding to max turns:

1) Stretch the motor as far as you dare,

2) wind in about half the expected turns at full stretch and

3) wind in the same number again while gradually relaxing the motor to the final length.

Assuming that the final motor length is equal to the unstretched length, a winding sequence in which the first half of the turns remain as twist while the second half convert directly to writhe loops at constant total twist would generate the proposed Tw = Wr structure.

Comments on the first draft of this note by some experienced rubber flyers revealed that various refinements and modifications of the basic rule-of-thumb winding process are practiced in the field. One suggestion, which makes a lot of sense, is to let the motor shorten a little during the first stage of winding to offset the increasing tension. In light of this insight, steps 1) and 2) could be changed to "stretch the motor to the maximum safe tension force" and "wind in about half the expected turns at constant tension force."

The reviewers also reported that there are significant differences in the degree of stretch and the relative number of turns used in step 1) by individual flyers. Some indoor flyers use a more complex process in which the motor is repeatedly stretched close to the limit and then wound back in.

A parameter worthy of special consideration is the final length of the wound motor as it is installed in the airframe. In modern commercial F1B's (Wakefields), the wound motor is reportedly mounted at a length L1 10-20% greater than the original unstretched length L. By contrast, scale fliers often use motors with L1 several times greater than the (sometimes limited) hook-to-peg space available in the model. Interestingly, the constant-volume geometrical model implies that there is more space for turns as the wound motor is stretched longer than L. Conversely, there less space for turns if the motor is allowed to contract to a length less than L. This is easy to see in the length-ratio calculation of max turns since, as the motor gets shorter it also gets fatter and the average wrapping circumference pR1 increases while xL remains the same. The performance implications of this feature of the constant-volume model remain to be explored.

Finally, there is no a priori reason to think that the optimum twist-to-writhe ratio is exactly 1:1. It may well be that differing distributions of twist and writhe giving improved energy performance can be obtained by stretching more or less, putting in more or fewer turns in the stretched state and/or mounting the motor at a wound length greater or less than the unstretched length L. Those questions will be left to the flying field and the test stand for now.

Conclusions

A very simple geometrical model of a stretched and spooled rubber motor with zero twist correctly predicts the dependence of max turns on motor cross-section but underestimates the total turns by a factor of about two. The discrepancy strongly suggests that the real motor structure near max turns contains a twist component about equal to the writhe component. Refinement of the model to include a twist component equal to the calculated writhe turns produces good agreement with published motor test results. The suggested Tw = Wr structure for a motor near max turns is consistent with the common practice of stretch winding to achieve maximum performance.

References

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(2) Calugareanu, Sur les classes d'isotropie des noeuds tridimensionnels et leurs invariants, *Czechoslovak Math. J.* **11**, 1961.

(3) R. Ricca, The energy spectrum of a twisted flexible string under elastic relaxation, *Journal of Physics A: Math. Gen.* **28** 1995.

(4) J. Barker, Rubber Motor Turns, *Free Flight News*, Farnborough England, March 1995.

(5) http://www.hippocketaeronautics.com/hpa_forum/index.php?topic=3464.0

(6) http://www.indoorduration.com/INAVWinding.htm

(7) Carrol Allen, private communication